

# Fault-Tolerant Approximate Shortest-Path Trees<sup>\*</sup>

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**Abstract.** The resiliency of a network is its ability to remain *effectively* functioning also when any of its nodes or links fails. However, to reduce operational and set-up costs, a network should be small in size, and this conflicts with the requirement of being resilient. In this paper we address this trade-off for the prominent case of the *broadcasting* routing scheme, and we build efficient (i.e., sparse and fast) *fault-tolerant approximate shortest-path trees*, for both the edge and vertex *single-failure* case. In particular, for an  $n$ -vertex non-negatively weighted graph, and for any constant  $\varepsilon > 0$ , we design two structures of size  $O(\frac{n \log n}{\varepsilon^2})$  which guarantee  $(1 + \varepsilon)$ -stretched paths from the selected source also in the presence of an edge/vertex failure. This favorably compares with the currently best known solutions, which are for the edge-failure case of size  $O(n)$  and stretch factor 3, and for the vertex-failure case of size  $O(n \log n)$  and stretch factor 3. Moreover, we also focus on the unweighted case, and we prove that an ordinary  $(\alpha, \beta)$ -spanner can be slightly augmented in order to build efficient fault-tolerant approximate *breadth-first-search trees*.

## 1 Introduction

Broadcasting a message from a source node to every other node of a network is one of the most basic communication primitives. Since this operation should be performed by making use of a both sparse and fast infrastructure, the natural solution is to root at the source node a *shortest-path tree* (SPT) of the underlying graph. However, the SPT, as any tree-based network topology, is highly sensitive to a link/node malfunctioning, which will unavoidably cause the disconnection of a subset of nodes from the source.

To be readily prepared to react to any possible (transient) failure in a SPT, one has then to enrich the tree by adding to it a set of edges selected from the underlying graph, so that the resulting structure will be 2-edge/vertex-connected

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w.r.t. the source. Thus, after an edge/vertex failure, these edges will be used to build up the alternative paths emanating from the root, each one of them in replacement of a corresponding original shortest path which was affected by the failure. However, if these paths are constrained to be *shortest*, then it can be easily seen that for a non-negatively real weighted and undirected graph  $G$  of  $n$  nodes and  $m$  edges, this may require as much as  $\Theta(m)$  additional edges, also in the case in which  $m = \Theta(n^2)$ . In other words, the set-up costs of the strengthened network may become unaffordable. Thus, a reasonable compromise is that of building a *sparse* and *fault-tolerant* structure which *accurately approximates* the shortest paths from the source, i.e., that contains paths which are longer than the corresponding shortest paths by at most a multiplicative *stretch* factor, for any possible edge/vertex failure. The aim of this paper is to show that very efficient structures of this sort do actually exist.

*Related work.* Let  $s$  denote a distinguished source vertex of a non-negatively real weighted and undirected graph  $G = (V(G), E(G))$ . We say that a spanning subgraph  $H$  of  $G$  is an *Edge-fault-tolerant  $\alpha$ -Approximate SPT* (in short,  $\alpha$ -EASPT), with  $\alpha > 1$ , if it satisfies the following condition: For each edge  $e \in E(G)$ , all the distances from  $s$  in the subgraph  $H - e = (V(H), E(H) \setminus \{e\})$  are  $\alpha$ -stretched w.r.t. the corresponding distances in  $G - e$ . When *vertex failures* are considered, then the EASPT is correspondingly called VASPT.

Our work is inspired by the paper of Parter and Peleg [13], which were concerned with the same problem but on *unweighted* graphs (and so they were focusing on the construction of an *edge-fault-tolerant  $\alpha$ -approximate Breadth-First Search tree* (in short,  $\alpha$ -EABFS). In that paper the authors present a 3-EABFS having at most  $4n$  edges.<sup>1</sup> Moreover, the authors also present a set of lower and upper bounds to the size of  $(\alpha, \beta)$ -EABFS, i.e., edge-fault-tolerant structures for which the length of a path is stretched by at most a factor of  $\alpha$  plus an additive term of  $\beta$ . Finally, assuming at most  $f = O(1)$  edge failures can take place, they show the existence of a  $(3(f+1), (f+1)\log n)$ -EABFS of size  $O(fn)$ .

On the other hand, if one wants to have an *exact* edge-fault-tolerant SPT (say ESPT), then as we said before this may require  $\Theta(n^2)$  edges. This is now in contrast with the unweighted case, where it can be shown the existence (see [12]) of an *edge/vertex-fault-tolerant BFS* (say EBFS/VBFS) of size  $O(n \cdot \min\{ecc(s), \sqrt{n}\})$ , where  $ecc(s)$  denotes the eccentricity of  $s$  in  $G$ . In the same paper, the authors also exhibit a corresponding lower bound of  $\Omega(n^{3/2})$  for the size of a EBFS. Moreover, they also treat the *multisource* case, i.e., that in which we look for a structure which incorporates an EBFS rooted at each vertex of a set  $S \subseteq V(G)$ . For this, they show the existence of a solution of size  $O(\sqrt{|S|} \cdot n^{3/2})$ , which is tight. Finally, the authors provide an  $O(\log n)$ -approximation algorithm for constructing an optimal (in terms of size) EBFS (also for the multisource case), and they show this is tight.

<sup>1</sup> Notice that this result is obtained through a rather involved algorithm that suitably enriches a BFS of  $G$  rooted at the source node, but, as we will point out in more detail later, a 3-EASPT of size at most  $2n$  (and then, *a fortiori*, a 3-EABFS of the same size), can actually be obtained as a by-product of the results given in [11].

As far as the vertex-failure problem is concerned, in [3] the authors study the related problem of computing *distance sensitivity oracles* (DSO) structures. Designing an efficient DSO means to compute, with a *low* preprocessing time, a *compact* data structure which is functional to *quickly* answer to some distance query following a component failure. Classically, DSO cope with single edge/vertex failures, and they have to answer to a point-to-point post-failure (approximate) distance query, or they have to report a point-to-point replacement short(est) path. In particular, in [3] the vertex-failure case w.r.t. a SPT is analyzed, and the authors compute in  $O(m \log n + n^2 \log n)$  time a DSO of size  $O(n \log n)$ , that returns a 3-stretched replacement path in time proportional to the path's size. As the authors specify in the paper, this DSO can be used to build a 3-VASPT of size  $O(n \log n)$ , and a  $(1 + \varepsilon)$ -VABFS of size  $O(\frac{n}{\varepsilon^3} + n \log n)$ . Actually, we point out that the latter structure can be easily sparsified so as to obtain a  $(1 + \varepsilon)$ -EABFS of size  $O(\frac{n}{\varepsilon^3})$ : in fact, its  $O(n \log n)$  size term is associated with an auxiliary substructure that, in the case of edge failures, can be made of linear size. This result is of independent interest, since it qualifies itself as the best current solution for the EABFS problem.

*Our results.* Our main result is the construction in polynomial time<sup>2</sup> of a  $(1 + \varepsilon)$ -VASPT of size  $O(\frac{n \log n}{\varepsilon^2})$ , for any  $\varepsilon > 0$ . This substantially improves on the 3-VASPT of size  $O(n \log n)$  given in [3]. To obtain our result, we perform a careful selection of edges that will be added to an initial SPT. The somewhat surprising outcome of our approach is that if we accept to have slightly stretched fault-tolerant paths, then we can drastically reduce the  $\Theta(n^2)$  size of the structure that we would have to pay for having fault-tolerant *shortest* paths! Actually, the analysis of the stretch factor and of the structure's size induced by our algorithm is quite involved. Thus, for clarity of presentation, we give our result in two steps: first, we show an approach to build a  $(1 + \varepsilon)$ -EASPT of size  $O(\frac{n \log n}{\varepsilon^2})$ , then we outline how this approach can be extended to the vertex-failure case.

Furthermore, we also focus on the unweighted case, and we exhibit an interesting connection between a fault-tolerant BFS and an  $(\alpha, \beta)$ -spanner. An  $(\alpha, \beta)$ -spanner of a graph  $G$  is a spanning subgraph  $H$  of  $G$  such that *all* the intra-node distances in  $H$  are stretched by at most a multiplicative factor of  $\alpha$  and an additive term of  $\beta$  w.r.t. the corresponding distances in  $G$ . We show how an ordinary  $(\alpha, \beta)$ -spanner of size  $\sigma = \sigma(n, m)$  can be used to build in polynomial time an  $(\alpha, \beta)$ -EABFS and an  $(\alpha, \beta)$ -VABFS of size  $O(\sigma)$  and  $O(\sigma + n \log n)$ , respectively. As a consequence, the EABFS problem is easier than the corresponding (non fault-tolerant) spanner problem, and we regard this as an interesting hardness characterization. Notice also that for all the significant values of  $\alpha$  and  $\beta$ , the size of an  $(\alpha, \beta)$ -spanner is  $\omega(n \log n)$ , which essentially means that the VABFS problem is easier than the corresponding spanner problem as well. This bridge between the two problems is useful for building sparse  $(1, \beta)$ -VABFS structures

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<sup>2</sup> We do not insist on the time efficiency in building our structures, since the focus of our paper, consistently with the literature, is on the trade-off between their size and their stretch factor.

by making use of the vast literature on additive  $(1, \beta)$ -spanners. For instance, the  $(1, 4)$ -spanner of size  $O(n^{\frac{7}{5}} \text{polylog}(n))$  given in [6], and the  $(1, 6)$ -spanner of size  $O(n^{\frac{4}{3}})$  given in [2], can be used to build corresponding vertex-fault-tolerant structures. Another interesting implication arises for the multisource **EABFS** problem. Indeed, given a set of multiple sources  $S \subseteq V(G)$ , the  $(\alpha, \beta)$ -spanner of size  $\sigma$  can be used to build a multisource  $(\alpha, \beta)$ -**EABFS** of size  $O(n \cdot |S| + \sigma)$ . This allows to improve, for  $|S| = \omega(n^{\frac{1}{15}} \text{polylog}(n))$ , the multisource  $(1, 4)$ -**EABFS** of size  $O(n^{\frac{4}{3}} \cdot |S|)$  given in [13]: indeed, it suffices to plug-in in our method the  $(1, 4)$ -spanner of size  $O(n^{\frac{7}{5}} \text{polylog}(n))$  given in [6].

*Other related results.* Besides fault-tolerant (approximate) SPT and BFS, there is a large body of literature on fault-tolerant short(est) paths in graphs. A natural counterpart of the structures considered in this paper, as we have seen before, are the DSO. For recent achievements on DSO, we refer the reader to [4, 8], and more in particular to [3, 10], where single-source distances are considered. Another setting which is very close in spirit to ours is that of *fault-tolerant spanners*. In [7], for weighted graphs and any integer  $k \geq 1$ , the authors present a  $(2k - 1, 0)$ -spanner resilient to  $f$  vertex (resp., edge) failures of size  $O(f^2 \cdot k^{f+1} \cdot n^{1+1/k} \cdot \log^{1-1/k} n)$  (resp.,  $O(f \cdot n^{1+1/k})$ ). This was later improved through a randomized construction in [9]. On the other hand, for the unweighted case, in [5] the authors present a general result for building a  $(1, O(f \cdot (\alpha + \beta)))$ -spanner resilient to  $f$  edge failures, by unioning an ordinary  $(1, \beta)$ -spanner with a fault-tolerant  $(\alpha, 0)$ -spanner resilient against up to  $f$  edge faults. Finally, we mention that in [1] it was introduced the resembling concept of *resilient spanners*, i.e., spanners such that whenever any edge in  $G$  fails, then the relative distance increases in the spanner are very close to those in  $G$ , and it was shown how to build a resilient spanner by augmenting an ordinary spanner.

## 2 Notation

We start by introducing our notation. For the sake of brevity, we give it for the case of edge failures, but it can be naturally extended to the node failure case.

Given a non-negatively real weighted, undirected, and 2-edge-connected graph  $G$ , we will denote by  $w_G(e)$  or  $w_G(u, v)$  the weight of the edge  $e = (u, v) \in E(G)$ . We also define  $w(G) = \sum_{e \in E(G)} w(e)$ . Given an edge  $e = (u, v)$ , we denote by  $G - e$  or  $G - (u, v)$  (resp.,  $G + e$  or  $G + (u, v)$ ) the graph obtained from  $G$  by removing (resp., adding) the edge  $e$ . Similarly, for a set  $F$  of edges,  $G - F$  (resp.,  $G + F$ ) will denote the graph obtained from  $G$  by removing (resp., adding) the edges in  $F$ .

We will call  $\pi_G(x, y)$  a shortest path between two vertices  $x, y \in V(G)$ ,  $d_G(x, y)$  its (weighted) length, and  $T_G(s)$  a SPT of  $G$  rooted at  $s$ . Whenever the graph  $G$  and/or the vertex  $s$  are clear from the context, we might omit them, i.e., we will write  $\pi(u)$  and  $d(u)$  instead of  $\pi_G(s, u)$  and  $d_G(s, u)$ , respectively. When considering an edge  $(x, y)$  of an SPT we will assume  $x$  and  $y$  to be the closest and the furthest endpoints from  $s$ , respectively.

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**Algorithm 1:** Algorithm for building an  $(1 + \varepsilon)$ -EASPT

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**Input** : A graph  $G$ ,  $s \in V(G)$ ,  $\varepsilon > 0$   
**Output:** A  $(1 + \varepsilon)$ -EASPT of  $G$  rooted at  $s$

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1  $H \leftarrow$  compute a 3-EASPT of size  $O(n)$  using the algorithm in Sect. 3.1.1 of [11].
2 for  $e \in E(T_G(s))$  in preorder w.r.t.  $T_G(s)$  do
3   for  $t \in V(G)$  in preorder w.r.t.  $T_G^{-e}(s)$  do
4     if  $d_H^{-e}(t) > (1 + \varepsilon)d_G^{-e}(t)$  then /* vertex  $t$  is bad for edge  $e$  */
5       Select a set of edges  $S \subseteq E(\pi_G^{-e}(t))$  (see details after Lemma 1)
6        $H \leftarrow H + S$ 
7 return  $H$ 
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Given an edge  $e \in E(G)$ , we define  $\pi_G^{-e}(x, y)$ ,  $d_G^{-e}(x, y)$  and  $T_G^{-e}(s)$  to be, respectively, a shortest path between  $x$  and  $y$ , its length, and a SPT in the graph  $G - e$ . Moreover, if  $P$  is a path from  $x$  to  $y$  and  $Q$  is a path from  $y$  to  $z$ , with  $x, y, z \in V(G)$ , we will denote by  $P \circ Q$  the path from  $x$  to  $z$  obtained by concatenating  $P$  and  $Q$ .

Given  $G$ , a vertex  $s \in V(G)$ , and an edge  $e = (u, v) \in E(T_G(s))$ , we denote by  $U_G(e)$  and  $D_G(e)$  the partition of  $V(G)$  induced by the two connected components of  $T(G) - e$ , such that  $U_G(e)$  contains  $s$  and  $u$ , and  $D_G(e)$  contains  $v$ . Then,  $C_G(e) = \{(x, y) \in E(G) : x \in U_G(e), y \in D_G(e)\}$  will denote the *cutset* of  $e$ , i.e., the set of edges crossing the cut  $(U_G(e), D_G(e))$ .

For the sake of simplicity we consider only edge weights that are strictly positive. However our entire analysis also extends to non-negative weights. Throughout the rest of the paper we will assume that, when multiple shortest paths exist, ties will be broken in a consistent manner. In particular we fix a SPT  $T = T_G(s)$  of  $G$  and, given a graph  $H \subseteq G$  and  $x, y \in V(H)$ , whenever we compute the path  $\pi_H(x, y)$  and ties arise, we will prefer the edges in  $E(T)$ . We will also assume that if we are considering a shortest path  $\pi_H(x, y)$  between  $x$  and  $y$  passing through vertices  $x'$  and  $y'$ , then  $\pi_H(x', y') \subseteq \pi_H(x, y)$ .

### 3 A $(1 + \varepsilon)$ -EASPT structure

First, we give a high-level description of our algorithm for computing a  $(1 + \varepsilon)$ -EASPT (see Algorithm 1). We build our structure, say  $H$ , by starting from an SPT  $T$  rooted at  $s$  which is suitably augmented with at most  $n - 1$  edges in order to make it become a 3-EASPT. Then, we enrich  $H$  incrementally by considering the tree edge failures in preorder, and by checking the disconnected vertices. When an edge  $e$  fails and a vertex  $t$  happens to be too stretched in  $H - e$  w.r.t. its distance from  $s$  in  $G - e$ , we add a suitable subset of edges to  $H$ , selected from the new shortest path to  $t$ . This is done so that we not only adjust the distance of  $t$ , but we also improve the stretch factor of a *subset* of its predecessors. This is exactly the key for the efficiency of our method, since altogether, up to a logarithmic factor, we maintain constant in an amortized sense the ratio between the size of the set of added edges and the overall distance improvement.

Let us now provide a detailed description of our algorithm. To build the initial 3-EASPT, it augments  $T$  by making use of a *swap algorithm* devised in [11]. More precisely, in that paper the authors were concerned with the problem of reconnecting in a best possible way (w.r.t. to a set of distance criteria) the two subtrees of an SPT undergoing an edge failure, through a careful selection of a *swap edge*, i.e., an edge with an endvertex in each of the two subtrees. In particular, they show that if we select as a swap edge for  $e = (u, v)$  – with  $u$  closer to the source  $s$  than  $v$  – the edge that lies on a shortest path in  $G - e$  from  $s$  to  $v$ , then the distances from the source towards all the disconnected vertices is stretched at most by a factor of 3.<sup>3</sup> Therefore, a 3-EASPT of size at most  $2n$  can be obtained by simply adding to a SPT rooted at  $s$  a such swap edge for each corresponding tree edge, and interestingly this improves the 3-EASPT of size at most  $4n$  provided in [13].

Then, our algorithm works in  $n - 1$  *phases*, where each phase considers an edge of  $T$  w.r.t. to a fixed preorder of the edges, say  $e_1, \dots, e_{n-1}$ . In the  $h$ -th phase, the algorithm considers the failure of  $e_h$ , and when a vertex  $t$  happens to be too stretched in  $H$  w.r.t.  $d^{-e_h}(t)$ , then we say that  $t$  is *bad* for  $e_h$  and we add a suitable subset  $S$  of edges to  $H$ . These edges are selected from  $\pi^{-e_h}(t)$  and they always include the last edge of  $\pi^{-e_h}(t)$ . We now show that this suffices to prove the correctness of the algorithm:

**Lemma 1.** *The structure  $H$  returned by the algorithm is a  $(1 + \varepsilon)$ -EASPT.*

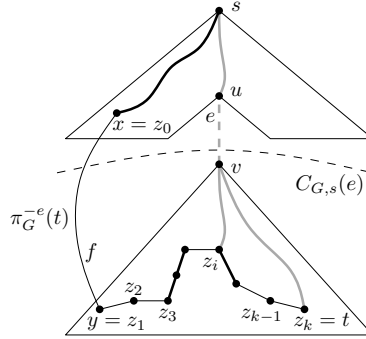
*Proof.* Let  $\tilde{H}$  be the structure built by the algorithm just before a bad vertex  $t$  for an edge  $e_h$  is considered. Assume by induction that, for every vertex  $z$  in  $T_G^{-e_h}(s)$  already considered in phase  $h$ , we have  $d_{\tilde{H}}^{-e_h}(z) \leq (1 + \varepsilon)d^{-e_h}(z)$ . Let  $f = (z, t)$  be the last edge of  $\pi^{-e_h}(t)$  and recall that  $f$  is always added to  $\tilde{H}$ . Hence we have:

$$\begin{aligned} d_{\tilde{H}}^{-e_h}(t) &\leq d_{\tilde{H}+f}^{-e_h}(t) \leq d_{\tilde{H}}^{-e_h}(z) + w(f) \leq (1 + \varepsilon)d^{-e_h}(z) + w(f) \\ &\leq (1 + \varepsilon)(d^{-e_h}(z) + d^{-e_h}(z, t)) = (1 + \varepsilon)d^{-e_h}(t). \quad \square \end{aligned}$$

It remains to describe the edge selection process and to analyze the size of our final structure. Let  $H_0$  be the initial 3-EASPT structure. Let us fix the failed edge  $e = (u, v)$  and a single bad vertex  $t$  for  $e$ . We call  $H'$  the structure built by the algorithm just before  $t$  is considered. Let  $f = (x, y)$  be the unique edge in  $C_G(e) \cap E(\pi_G^{-e}(t))$ . Consider the subpath of  $\pi_G^{-e}(t)$  going from  $x$  to  $t$  and let  $x_0, x_1, \dots, x_r$  be its vertices, in order. We consider the set  $Z = \{x_i : (x_{i-1}, x_i) \notin E(H_0)\}$ , we name its vertices  $z_1, \dots, z_k$  with  $k = |Z| - 1$ , in order and we let  $z_0 = x$  (see Figure 1). We define  $\alpha_i = \frac{d_{H'}^{-e}(z_i)}{d^{-e}(z_i)}$ . It follows from the definitions and from Lemma 1 that we have  $\alpha_0 = 1$ ,  $\alpha_j \leq (1 + \varepsilon)$  for  $1 \leq j < k$  and  $\alpha_k > 1 + \varepsilon$ .

Think of the edges in  $\pi^{-e}(t)$  as being directed towards  $t$  for a moment. In the following we will describe how to select the set  $S$  of edges used by the algorithm.

<sup>3</sup> Actually, in [11] it is not explicitly claimed the 3-stretch factor, but this is implicitly obtained by the qualitative analysis of the swap procedure therein provided.



**Fig. 1.** Edge selection phase of Algorithm 1 when a bad vertex  $t$  for the failing edge  $e$  is considered. Bold edges belong to  $H_0$  while the black path is  $\pi_G^{-e}(t)$ .

In particular, we will select  $\eta \geq 1$  edges entering into the last  $\eta$  vertices in  $Z$ . This choice of  $S$  will ensure that the overall decrease of the values  $\alpha_i$  in  $H' + S$  will be at least  $\frac{\varepsilon}{\mathcal{H}_n}\eta$  where  $\mathcal{H}_n$  denotes the  $n$ -th *harmonic number*.

We exploit the fact that, after adding the set  $S$ , each “new value”  $\alpha_i$  with  $i > k - \eta = j$ , will not be larger than  $\alpha_j$  as we will show in the following.

Consider the sequence  $\gamma_0, \dots, \gamma_k$  where  $\gamma_i = 1 + \frac{\varepsilon}{\mathcal{H}_k}(\mathcal{H}_k - \mathcal{H}_{k-i})$ . Notice that the sequence is monotonically increasing from  $\gamma_0 = 1$  to  $\gamma_k = 1 + \varepsilon$ . Let  $0 \leq j < k$  be the largest index such that  $\alpha_j \leq \gamma_j$ . Notice that  $j$  always exists as  $\alpha_0 = \gamma_0$  and that  $\alpha_k > \gamma_k$ . We set  $\eta = k - j$  so that the set  $S$  is defined accordingly. Let  $U = \{z_{j+1}, \dots, z_k\}$  be the set of vertices for which an incoming edges has been added in  $S$ .

For every vertex  $z \in U$  we define the following path in  $H' + S$ :  $P(z) = \pi_{H'}^{-e}(z_j) \circ \pi(z_j, z)$ . Notice that  $\pi(z_j, z)$  is entirely contained in  $H' + S$ . We define  $\alpha'_i = \frac{w(P(z_i))}{d^{-e}(z_i)}$ , and note that  $\alpha'_i$  is an upper bound to the stretch of  $z$  in  $H' + S$ .

**Lemma 2.** For  $i > j$ ,  $\alpha'_i \leq \alpha_j < \alpha_i$ .

*Proof.* By definition of  $j$ , we have  $\alpha_j \leq \gamma_j < \gamma_i < \alpha_i$ . Now we prove  $\alpha'_i \leq \alpha_j$ :

$$\alpha'_i = \frac{w(P(z))}{d^{-e}(z_i)} = \frac{d_{H'}^{-e}(z_j) + d(z_j, z_i)}{d^{-e}(z_i)} \leq \frac{\alpha_j d^{-e}(z_j) + d^{-e}(z_j, z_i)}{d^{-e}(z_i)} \leq \frac{\alpha_j d^{-e}(z_i)}{d^{-e}(z_i)} = \alpha_j.$$

□

We now lower-bound the overall decrease of the values  $\alpha'_i$ 's w.r.t. the corresponding  $\alpha_i$ 's by using the following inequalities:

$$\begin{aligned} \sum_{z \in U} \left( \frac{d_{H'}^{-e}(z)}{d^{-e}(z)} - \frac{w(P(z))}{d^{-e}(z)} \right) &= \sum_{i=j+1}^k (\alpha_i - \alpha'_i) \geq \sum_{i=j+1}^k (\alpha_i - \alpha_j) \geq \sum_{i=j+1}^k (\gamma_i - \gamma_j) \\ &= \frac{\varepsilon}{\mathcal{H}_k} \sum_{i=j+1}^k (\mathcal{H}_{k-j} - \mathcal{H}_{k-i}) = \frac{\varepsilon}{\mathcal{H}_k} (k - j) \geq \frac{\varepsilon}{\mathcal{H}_n} \eta. \end{aligned}$$

where in the last but one step we used the well-known equality that for every  $j \leq k$ ,  $\sum_{i=j+1}^k (\mathcal{H}_{k-j} - \mathcal{H}_{k-i}) = k - j$ .

The above selection procedure is repeated by the algorithm for every failed edge  $e_h$  and for every corresponding bad vertex. We now focus on the  $h$ -th phase of the algorithm. Let  $U_h$  be the union of all the sets  $U$  used when considering the bad vertices of the phase  $h$ . Moreover let  $V_h = \bigcup_{i=1}^h U_i$  and notice that  $V_0 = \emptyset$ . For a vertex  $z \in U_h$ , let  $P_h(z)$  be the *last* path  $P(z)$  built by the algorithm, as defined above. Let  $H_h$  (resp.,  $H'_h$ ) be the structure built by the algorithm at the end (resp., start) of the phase  $h$  and let  $m_h$  be the number of new edges added during the phase  $h$ . By summing over all the bad vertices for edge  $e_h$ , we have:

$$\textbf{Lemma 3.} \quad \sum_{z \in U_h} \left( \frac{d_{H'_h}^{-e_h}(z)}{d^{-e_h}(z)} - \frac{w(P_h(z))}{d^{-e_h}(z)} \right) \geq m_h \frac{\varepsilon}{\mathcal{H}_n}.$$

Now, let us define a function  $\phi_h(z)$  for every  $z \in V$ :

$$\phi_h(z) = \begin{cases} 0 & \text{if } z \notin V_h \\ w(P_h(z)) & \text{if } z \in U_h \\ \phi_{h-1}(z) & \text{if } z \in V_h \setminus U_h \end{cases}$$

The proofs of next three lemmas are postponed to the full version of the paper.

**Lemma 4.** *For every  $z \in U_h$  we have  $d_G^{-e_h}(z) < \frac{2}{\varepsilon} d_G(z)$ .*

**Lemma 5.** *For  $z \in V_{h-1}$ ,  $\phi_{h-1}(z) \geq d_{H_h}^{-e_h}(z)$ .*

**Lemma 6.** *For  $z \in U_h$ ,  $d_{H'_h}^{-e_h}(z) \geq w(P_h(z))$ .*

We now prove the following:

$$\textbf{Lemma 7.} \quad \sum_{z \in U_h} \frac{\phi_{h-1}(z)}{d(z)} - \sum_{z \in U_h} \frac{\phi_h(z)}{d(z)} \geq m_h \frac{\varepsilon}{\mathcal{H}_n} - |U_h \setminus V_{h-1}| \frac{6}{\varepsilon}.$$

*Proof.* By Lemmas 3–6, and since the initial structure  $H_0$  is a 3-EASPT, we have:

$$\begin{aligned} & \sum_{z \in U_h} \frac{\phi_{h-1}(z)}{d(z)} - \sum_{z \in U_h} \frac{\phi_h(z)}{d(z)} \geq \sum_{z \in U_h \cap V_{h-1}} \left( \frac{\phi_{h-1}(z)}{d(z)} - \frac{\phi_h(z)}{d(z)} \right) + \sum_{z \in U_h \setminus V_{h-1}} \left( \frac{\phi_{h-1}(z)}{d(z)} - \frac{\phi_h(z)}{d(z)} \right) \\ & \geq \sum_{z \in U_h \cap V_{h-1}} \left( \frac{d_{H'_h}^{-e_h}(z)}{d(z)} - \frac{\phi_h^{-e}(z)}{d(z)} \right) + \sum_{z \in U_h \setminus V_{h-1}} -\frac{\phi_h(z)}{d(z)} \\ & = \sum_{z \in U_h \cap V_{h-1}} \left( \frac{d_{H'_h}^{-e_h}(z)}{d(z)} - \frac{w(P_h(z))}{d(z)} \right) + \sum_{z \in U_h \setminus V_{h-1}} \left( \frac{d_{H'_h}^{-e_h}(z)}{d(z)} - \frac{w(P_h(z))}{d(z)} \right) - \sum_{z \in U_h \setminus V_{h-1}} \frac{d_{H'_h}^{-e_h}(z)}{d(z)} \\ & \geq \sum_{z \in U_h \cap V_{h-1}} \left( \frac{d_{H'_h}^{-e_h}(z)}{d^{-e_h}(z)} - \frac{w(P_h(z))}{d^{-e_h}(z)} \right) + \sum_{z \in U_h \setminus V_{h-1}} \left( \frac{d_{H'_h}^{-e_h}(z)}{d^{-e_h}(z)} - \frac{w(P_h(z))}{d^{-e_h}(z)} \right) - \sum_{z \in U_h \setminus V_{h-1}} \frac{3d^{-e_h}(z)}{d(z)} \\ & \geq \sum_{z \in U_h} \left( \frac{d_{H'_h}^{-e_h}(z)}{d^{-e_h}(z)} - \frac{w(P_h(z))}{d^{-e_h}(z)} \right) - \sum_{z \in U_h \setminus V_{h-1}} \frac{3d^{-e_h}(z)}{d(z)} \geq m_h \frac{\varepsilon}{\mathcal{H}_n} - |U_h \setminus V_{h-1}| \frac{6}{\varepsilon}. \quad \square \end{aligned}$$



We now define a global potential function  $\Phi$ :

$$\Phi(h) = \sum_{z \in V_h} \frac{\phi_h(z)}{d(z)}$$

for  $0 < h \leq n-1$ . Notice that we trivially have  $\Phi(h) \geq 0$ .

**Theorem 1.** *The structure  $H$  returned by the algorithm is a  $(1 + \varepsilon)$ -EASPT of size  $O(\frac{n \log n}{\varepsilon^2})$ .*

*Proof.* The fact that  $H$  is a  $(1 + \varepsilon)$ -EASPT follows from Lemma 1. Concerning the size of  $H$ , since  $H_0$  contains  $O(n)$  edges, we only focus on bounding the number  $\mu = \sum_{h=1}^{n-1} m_h$  of edges in  $E(H) \setminus E(H_0)$ . Using Lemma 7, we can write:

$$\begin{aligned} \Phi(i) &= \sum_{z \in V_{i-1} \setminus U_i} \frac{\phi_i(z)}{d(z)} + \sum_{z \in U_i} \frac{\phi_i(z)}{d(z)} \\ &= \sum_{z \in V_{i-1} \setminus U_i} \frac{\phi_{i-1}(z)}{d(z)} + \sum_{z \in U_i} \frac{\phi_{i-1}(z)}{d(z)} - \left( \sum_{z \in U_i} \frac{\phi_{i-1}(z)}{d(z)} - \sum_{z \in U_i} \frac{\phi_i(z)}{d(z)} \right) \\ &\leq \sum_{z \in V_{i-1} \setminus U_i} \frac{\phi_{i-1}(z)}{d(z)} + \sum_{z \in V_{i-1} \cap U_i} \frac{\phi_{i-1}(z)}{d(z)} + \sum_{z_i \in U_i \setminus V_{i-1}} \frac{\phi_{i-1}(z)}{d(z)} - m_h \frac{\varepsilon}{\mathcal{H}_n} + |U_h \setminus V_{h-1}| \frac{6}{\varepsilon} \\ &\leq \Phi(i-1) + 0 - m_h \frac{\varepsilon}{\mathcal{H}_n} + |U_h \setminus V_{h-1}| \frac{6}{\varepsilon}. \end{aligned}$$

Unfolding the previous recurrence relation we obtain:

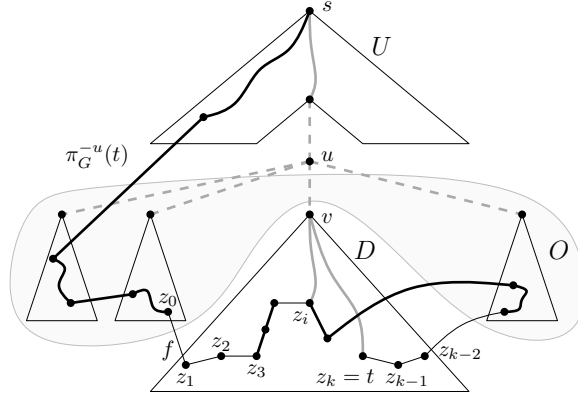
$$0 \leq \Phi(n-1) \leq |V_{n-1}| \frac{6}{\varepsilon} - \frac{\varepsilon}{\mathcal{H}_n} \sum_{h=0}^{n-1} m_h \leq n \frac{6}{\varepsilon} - \frac{\varepsilon}{\mathcal{H}_n} \mu$$

which we finally solve for  $\mu$  to get  $\mu = O(\frac{n \log n}{\varepsilon^2})$ .  $\square$

## 4 A $(1 + \varepsilon)$ -VASPT structure

In this section we extend our previous  $(1 + \varepsilon)$ -EASPT structure to deal with vertex failures. In order to do so we will build a different subgraph  $H_0$  having suitable properties that we will describe. Then we will use the natural extension of Algorithm 1 where we consider (in preorder) vertex failures instead of edge failures. We now describe the construction of  $H_0$  and then argue how the previous analysis can be adapted to show the same bound on the size of  $H$ .

The structure  $H_0$  is initially equal to  $T$  and it is augmented by using a technique similar to the one shown in [3]: the SPT  $T$  of  $G$  is suitably decomposed into ancestor-leaf vertex-disjoint paths. Then, for each path, an approximate structure is built. This structure will provide approximate distances towards any vertex of the graph when any vertex along the path fails. The union of  $T$  with all those structures will form  $H_0$ .



**Fig. 2.** Edge selection phase of the vertex-version of Algorithm 1 when a bad vertex  $t$  for the failing vertex  $u$  is considered. Bold edges belong to  $H_0$  while the black path is  $\pi_G^{-u}(t)$ . Notice that all  $z_i$ s belong to the down set  $D$ .

Fix a path  $Q$  of the previous decomposition starting from a vertex  $q$ , and let  $T_q$  be the subtree of  $T$  rooted at  $q$ . Moreover, let  $u \in V(Q)$  be a failing vertex, and let  $v$  be the next vertex in  $Q$ .<sup>4</sup> We partition the vertices of the forest  $T - u$  into three sets: (i) the *up set*  $U$  containing all the vertices of the tree rooted at  $s$ , (ii) the *down set*  $D$  containing all the vertices of the tree rooted at  $v$ , and (iii) the *others set*  $O$  containing all the remaining vertices (see Figure 2).

We want to select a set of edges to add to  $H$ . In order to do so, we construct a SPT  $T'$  of  $G - u$  and we imagine that its edges are directed towards the leaves. We select all the edges of  $E(T') \setminus E(T)$  that do not lead to a vertex in  $D$ , plus the unique edge of  $\pi^{-u}(v)$  that crosses the cut induced by the sets  $U \cup O$  and  $D$ . Notice that  $T - u$  contains all the paths in  $T'$  towards the vertices in  $U$ , and that each vertex has at most one incoming edge in  $T'$ . This implies that the number of selected edges is at most  $|O| + 1$ .

The above procedure is repeated for all the failing vertices of  $Q$ , in order. As the sets  $O$  associated with the different vertices are disjoint we have that, while processing  $Q$ , at most  $|V(T_q)| + |Q| = O(|V(T_q)|)$  edges are selected. We use the path decomposition described in [3] that can be recursively defined as follows: given a tree, we select a path  $Q$  from the root to a leaf such that the removal of  $Q$  splits the tree into a forest where the size of each subtree is at most half the size of the original tree. We then proceed recursively on each subtree. Using this approach, the size of the entire structure  $H_0$  can be shown to be  $O(n \log n)$  [3].

We now prove some useful properties of the structure  $H_0$ . First of all, observe that, by construction and similarly to the edge-failure case, we immediately have:

**Lemma 8.** *Consider a failed vertex  $u$  and another vertex  $z \neq u$ . We have: (i)  $d_{H_0}^{-u}(v) = d^{-u}(v)$ , and (ii) for  $z \in D$ , it holds  $d_{H_0}^{-u}(z) \leq 3d^{-u}(z)$ .*

<sup>4</sup> W.l.o.g. we are assuming that the failing vertex  $u$  is not a leaf, as otherwise  $T - u$  is already a SPT of  $G - u$ .

Moreover, we also have the following (proof postponed to the full version of the paper):

**Lemma 9.** *Consider a failed vertex  $u$ . During the execution of the vertex-version of Algorithm 1, every bad vertex  $t$  for  $u$  will be in  $D$ .*

At this point, the same analysis given for the case of edge failures can be retraced for vertex failures as well. We point out that Lemma 9 ensures that every bad vertex for  $u$  is in the same subtree as  $v$ . Also notice that all the vertices  $z_i$ 's are, by definition, in the same subtree as well (see Figure 2). The above, combined with Lemma 8 (i), is needed by the proof of Lemma 4, while Lemma 8 (ii) is used in the proof of Lemma 7. Hence we have:

**Theorem 2.** *The vertex-version of Algorithm 1 computes a  $(1 + \varepsilon)$ -VASPT of size  $O(\frac{n \log n}{\varepsilon^2})$ .*

## 5 Relation with $(\alpha, \beta)$ -spanners in unweighted graphs

In this section we turn our attention to the unweighted case, and we provide two polynomial-time algorithms that augment an  $(\alpha, \beta)$ -spanner of  $G$  so to obtain an  $(\alpha, \beta)$ -EABFS/VABFS. We present the algorithm for the vertex-failure case and show how it can be adapted to the edge-failure case.

The algorithm first augments the structure  $H_0$  computed so as explained in Section 4 and then adds its edges to the  $(\alpha, \beta)$ -spanner of  $G$ . The structure  $H_0$  is augmented as follows. The vertices of the BFS of  $G$  rooted at  $s$  are visited in preorder. Let  $u$  be the vertex visited by the algorithm and let  $D$  be the set of vertices of the tree defined so as explained in Section 4 w.r.t the path decomposition computed for  $H_0$ . For every  $t \in D$ , the algorithm checks whether  $\pi_G^{-u}(s, t)$  contains no vertex of  $D \setminus \{t\}$  and  $d_G^{-u}(s, t) < d_{H_0}^{-u}(s, t)$ . If this is the case, then the algorithm augments  $H_0$  with the edge of  $\pi_G^{-u}(s, t)$  incident to  $t$ .

The following observation is crucial to prove the algorithm correctness.

**Fact 1** *For every vertex  $u$  and every vertex  $t \in V(G) \setminus \{u\}$  such that  $\pi_G^{-u}(t)$  contains a vertex in  $D$ , let  $x$  and  $y$  be the first and last vertex of  $\pi_G^{-u}(t)$  that belong to  $D$ , respectively. We have  $d_{H_0}^{-u}(x) = d_G^{-u}(x)$  and  $d_{H_0}^{-u}(y, t) = d_G^{-u}(y, t)$ .*

We can now give the following (proof postponed to the full version of the paper):

**Theorem 3.** *Given an unweighted graph  $G$  with  $n$  vertices and  $m$  edges, a source vertex  $s \in V(G)$ , and an  $(\alpha, \beta)$ -spanner for  $G$  of size  $\sigma = \sigma(n, m)$ , the algorithm computes an  $(\alpha, \beta)$ -VABFS w.r.t.  $s$  of size  $O(\sigma + n \log n)$ .*

Now, we adapt the algorithm to prove a similar result for the  $(\alpha, \beta)$ -EABFS. The algorithm first augments a BFS tree  $T$  of  $G$  rooted at  $s$  and then adds its edges to the  $(\alpha, \beta)$ -spanner of  $G$ . The tree  $T$  is augmented by visiting its edges in preorder. Let  $e$  be the edge visited by the algorithm. For every  $t \in D_G(e)$ ,

the algorithm checks whether  $\pi_G^{-e}(s, t)$  contains no vertex of  $D_G(e) \setminus \{t\}$  and  $d_G^{-e}(s, t) < d_T^{-e}(s, t)$ . If this is the case, then the algorithm augments  $T$  with the edge of  $\pi_G^{-e}(s, t)$  incident to  $t$ . In the full version of the paper it will be shown that the proof of Theorem 3 can be adapted to prove the following:

**Theorem 4.** *Given an unweighted graph  $G$  with  $n$  vertices and  $m$  edges, a source vertex  $s \in V(G)$ , and an  $(\alpha, \beta)$ -spanner for  $G$  of size  $\sigma$ , the algorithm computes an  $(\alpha, \beta)$ -EABFS w.r.t.  $s$  of size less than or equal to  $\sigma + 3n$ .*

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